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FADDEEV-JACKIW APPROACH TO HIDDEN SYMMETRIES[1]

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Abstract

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The study of hidden symmetries within Dirac's formalism does not possess a systematic procedure due to the lack of first-class constraints to act as symmetry generators. On the other hand, in the Faddeev-Jackiw approach, gauge and reparametrization symmetries are generated by the null eigenvectors of the sympletic matrix and not by constraints, suggesting the possibility of dealing systematically with hidden symmetries throughout this formalism. It is shown in this paper that indeed hidden symmetries of noninvariant or gauge fixed systems are equally well described by null eigenvectors of the sympletic matrix, just as the explicit invariances. The Faddeev-Jackiw approach therefore provide a systematic algorithm for treating all sorts of symmetries in an unified way. This technique is illustrated here by the SL(2,R) affine Lie algebra of the 2-D induced gravity proposed by Polyakov, which is a hidden symmetry in the canonical approach of constrained systems via Dirac's method, after conformal and reparametrization invariances have been fixed.

1 Introduction

Invariant theories, both of gauge and reparametrization type, have been dealt with in the last 50 years by Dirac's Hamiltonian formalism[1], with its categorization of constraints as first and second-class and the separation of equalities as weak and strong. Within this context the symmetries of a given theory are generated by a complete set of first-class constraints obtained by a well defined algorithm, and these constraints are used to reduce the original Hilbert space into the subspace of physical solutions. Nevertheless, noninvariant theories and gauge-fixed invariant models presenting only second-class constraints may still possess a set of (hidden) symmetries that cannot be unveiled by Dirac's methodology. As we know, second-class constraints are used to alter the original (Poisson) bracket operation so that they can be imposed as operators identities, but cannot be used as symmetry generators. Within Dirac's formalism, the disclosure of those symmetries that may be (possibly) hidden in a model will depend then crucially on our ability and experience on the subject, due to the lack of a systematic methodology.

Recently, Faddeev and Jackiw[2] proposed an alternative approach to constrained systems that avoids the separation of constraints into first and second-class and the use of weak and strong equalities. This new method of analysis has been successfully used by many authors[3, 4, 5, 6, 7, 8], and is by now a standard technique to deal with constrained systems. In a series of papers [9, 10, 11, 12], this author and collaborators have proposed an algorithm, based on the Faddeev-Jackiw technique, to treat systems with constraints, both invariant and noninvariant, on the same foot: for noninvariant systems the sympletic matrix defining the geometrical structure of the theory will always end up being invertible even if it was not to begin with. This matrix inverse will then provide the Dirac brackets used in the quantization process(up to ordering ambiguities). For gauge theories, the sympletic matrix is never invertible, but it has been shown [11, 12] that the null eigenvectors of this matrix are the generators of the intrinsic gauge symmetry of the model.

It is our intention in the present work to show that those symmetries that persist after gauge fixing and are called hidden in the literature due to the lack of a set of first-class constraints to generate them, need no special treatment in the Faddeev-Jackiw context, being described by a set of null eigenvectors of the sympletic matrix exactly as for the explicit symmetries. While for those systems where Dirac's formalism is free of pathologies, the advantage of Faddeev-Jackiw's method over Dirac's seems to be only of practical nature (it is simpler and quicker to use), the case of hidden symmetries shows a definite edge in favor of the Faddeev-Jackiw geometric approach.

To illustrate this systematic process of finding the hidden symmetries of an arbitrary action, we study the SL(2,R) symmetry of Polyakov's 2D induced gravity. The present status of this theory is as follows. The SL(2,R) current algebra symmetry of the induced gravity was discovered by Polyakov [13] a few years ago. Later on this symmetry has led to an exact solution of the 2-D quantum theory of gravity interacting with conformal matter [14]. It allows us to write down recursion relations for the (Euclidean) Green functions and to determine the renormalization of the parameters of the theory. This symmetry, which is hidden under the more conventional approach to current algebra, was discovered by Polyakov by analyzing the anomaly equation. In spite of its importance for the development of this very active field, the physical origin of this symmetry remained obscure[4] (for a nice review on the development of this theory and a complete list of references see Ref.[15]).

The study of this symmetry under the canonical point of view has been pursued by many authors [16, 17, 18, 19], mostly based on Dirac's theory of constrained systems[1]. The puzzling point, as indicated by Egorian and Manvelian[17] and by Barcelos-Neto [19], is that the local form of Polyakov's model in the light cone gauge has only second-class constraints when written in Dirac's light-front coordinates, and no constraints whatsoever if written in instant-front variables[20]. Dirac's method for dealing with constraints is well established and the basic feature of this approach is that gauge and reparametrization symmetries are related with first-class constraints, not present in this model. In [17], the (hidden) SL(2,R) symmetry was found following a somewhat modified Dirac's method, but their solution was criticized in [19] because they did not find the (supposedly also hidden) first-class constraints that generate this symmetry. This task has been pursued in [19] more recently. Working with the theory compactified in the (light-cone) space-dimension (x^-) to a circle, the author was able to identify among the (infinite) set of second-class constraints, three apparently first-class constraints. He identified these constraints as the zero-modes of the set of second-class constraints of the gauge fixed theory. Unfortunately these constraints do not seem do correspond to the generators of the SL(2,R) symmetry, as mentioned by the author. This happens because these zero-mode constraints are in fact second-class, which explains why they fail to generate the required symmetry.

An alternative approach to the problem has appeared in the literature quite recently [21]. In this work, the authors were able to clearly identify the generators of this hidden symmetry as improper gauge constraints, as proposed by Regge and Teitelboim [22, 23, 24]. The separation of first-class constraints into proper and improper

[4] See note added at the end of the paper.

was proposed in [22] in order to resolve some difficulties of Dirac's method: sometimes, among the second-class constraints there appears to exist a subset of first-class constraints. In [23] it was shown that a precise mathematical treatment of this subset of first-class constraints reveals that they are in fact second-class and, if treated as such, Dirac's procedure may be consistently applied. Proper gauge transformations then, represent true symmetries of the theory and do not change the physical space of states. Improper gauge transformations, as defined in [22, 23] do change the physical state of the system, mapping one physical solution into another different one. While proper gauge transformations can be eliminated by gauge fixing, improper gauge symmetries cannot since that would exclude physically allowed solutions. Thus improper gauge transformations will remain as hidden symmetries.

The outline of this paper is as follows. In section II we review the FJ formalism for invariant theories and show that the explicit symmetries, as well as the hidden, are generated by the null eigenvectors of the sympletic matrix, giving in this way an unified treatment for the subject under this approach. We end up this section with a discussion of two simple examples of physical systems presenting hidden symmetries. We show next, in Section III, that the sympletic matrix for the Polyakov model in the light cone gauge is singular and possess three null eigenvectors that happen to be the generators of the SL(2,R) residual symmetry which, therefore, becomes manifest in the Faddeev-Jackiw formalism.

2 Faddeev-Jackiw Formalism For Invariant Theories

Faddeev and Jackiw[2] showed that the treatment of a Lagrangian that is first-order in time derivatives dispense the use Dirac's formalism. Consider a dynamical system with N bosonic degrees of freedom q_i described by the following Lagrangian[5]

$$L = a_i(q)\dot{q}_i - V(q)\ ; \quad i = 1, ..., N \tag{1}$$

$V(q)$ is the sympletic potential and $a_i(q)$ are the components of the canonical one-form $a(q) = a_i(q)dq_i$. This presentation of the theory is called non-standard since its equations of motion do not involve accelerations, and read

[5] We only consider a finite number of bosonic coordinates, the extension to the fermionic case [25] and to field theory being straightforward[9].

$$\bar{f}_{ij}\dot{q}_j - \frac{\partial V}{\partial q_i} = 0 \tag{2}$$

with the sympletic matrix $\bar{f}_{ij}$ being defined by

$$\bar{f}_{ij} = \frac{\partial a_j}{\partial q_i} - \frac{\partial a_i}{\partial q_j} \tag{3}$$

If this matrix is nonsingular, its inverse elements give the Dirac brackets of the theory (1), with the dynamics being described by

$$\{F, G\} = \frac{\partial F}{\partial q_i}(\bar{f}_{ij})^{-1}\frac{\partial G}{\partial q_j} \tag{4}$$

However, when $\bar{f}_{ij}$ is singular the scheme above no longer works due to the existence of true constraints that reduce the number of independent degrees of freedom of the theory. These (sympletic) constraints will appear in this formalism as algebraic relations $\Omega^a(q)$ needed to maintain the consistency of the equations of motion upon multiplication by the (left) zero-modes v_i^a of $\bar{f}_{ij}$. Then, from (2) we get

$$\Omega^a \equiv (v_i^a)^T \frac{\partial V}{\partial q_i} = 0 \tag{5}$$

where T stands for matrix transposition, and [6]

$$(v_i^a)^T \bar{f}_{ij} = 0 \ ; \quad a = 1, ..., M \tag{6}$$

define the M independent (left) zero-modes of $\bar{f}_{ij}$.

There are basically three situations that can arise out of the *scalar product* defining the sympletic constraints (5), which depends uniquely on the *angle* between $v_i^{(a)}$ and $\frac{\partial V}{\partial q_i}$ for each value of a. In the case of noninvariant systems all zero-modes are nonorthogonal to the gradient of the sympletic potential and constitute a true set of (sympletic) constraints that can be iteratively implemented into the canonical sector of the original theory (1), producing, step by step, a deformation of the original sympletic matrix, which end up being nonsingular[10]. In Dirac's terminology, this corresponds to a theory with purely second-class constraints. The sympletic algorithm described above has been shown to lead to the same results as that of Dirac's,

[6]For the case of field theory, where the zero-modes are in general operator valued objects, transposition is defined in the sense of $\int (v^T A)B \equiv \int A(vB)$.

although in a more economical way. A completely different situation appears when the gradient of the sympletic potential happens to be orthogonal to all zero-modes. Then condition (5) vanishes identically, the equations of motion are automatically validated and no sympletic constraints appear. This happens due to symmetries in the sympletic potential, a situation typical of gauge and reparametrization invariant theories. There is no hope to deform the sympletic matrix into an invertible one unless a gauge-fixing term is introduced to break the symmetry[11], or one has the constraint solved and pass, via Darboux's transformation, to a reduced canonical set[26]. Finally, we can have a mixed situation where some zero-modes are orthogonal to $\frac{\partial V}{\partial q_i}$ and some are not. In such cases one uses the sympletic algorithm to eliminate the constraints until only those zero-modes associated with symmetries remain.

Let us assume, for definiteness, that all sympletic constraints have been eliminated, and therefore only the zero-modes associated with gauge symmetries are still present. In first-order such theory can be written as

$$L = a_k(q)\dot{q}_k + \dot{\eta}_a \Omega^a(q) - V(q) \tag{7}$$

where the sympletic variables $q_k; k = 1, \ldots, N$ and $\eta_a; a = 1, \ldots, M$ form a set of gauge fields. We have also relabeled the Lagrange multipliers $\lambda_a \rightarrow -\dot{\eta}_a$[11]. By hypothesis, the sympletic matrix $(\bar{f}_{km})$ constructed out of the q_k-variables only is nonsingular, i.e., det $\bar{f} \neq 0$. In terms of the q's and η's the sympletic matrix becomes,

$$f = \begin{pmatrix} (\bar{f}) & (\frac{\partial \Omega}{\partial q}) \\ -(\frac{\partial \Omega}{\partial q})^T & 0 \end{pmatrix} \tag{8}$$

In this compact notation $(\bar{f})$ represents a (NxN) nonsingular matrix defined in (3). $(\frac{\partial \Omega}{\partial q})$ represents a rectangular (NxM) matrix defined as

$$\left(\frac{\partial \Omega}{\partial q}\right)_{ka} = \begin{pmatrix} \frac{\partial \Omega^1}{\partial q_1} & \frac{\partial \Omega^2}{\partial q_1} & \cdots & \frac{\partial \Omega^M}{\partial q_1} \\ \frac{\partial \Omega^1}{\partial q_2} & \frac{\partial \Omega^2}{\partial q_2} & \cdots & \frac{\partial \Omega^M}{\partial q_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \Omega^1}{\partial q_N} & \frac{\partial \Omega^2}{\partial q_N} & \cdots & \frac{\partial \Omega^M}{\partial q_N} \end{pmatrix} \tag{9}$$

The sympletic matrix (f) in (8) may have M zero-modes with the following (block) structure

$$(v_k^a) = \begin{pmatrix} -(\bar{f}_{km})^{-1} \frac{\partial \Omega^a}{\partial q_m} \\ 1^{(a)} \end{pmatrix} \tag{10}$$

where the first *element* is an (Nx1) column and $1^{(a)}$ is an (Mx1) column of zeros except in its a-th entry that is unity. For instance,

$$1^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{11}$$

Using (8) and (10), the zero-mode condition (6) can be written as two sets of equations; the first being automatically satisfied, while the second set is valid, as long as Ω^a defines an algebra under $\bar{f}$, i.e.,

$$\{\Omega^a, \Omega^b\} = C^{abc}\Omega^c \tag{12}$$

with the curly bracket operation being defined by (4). Therefore, the vectors (v_k^a) in (10) define a full set of null eigenvectors of the sympletic matrix (f) over the constraint surface. Since by hypothesis the gradient of the sympletic potential is orthogonal to all zero-modes, they must be the generators of the symmetry transformation that leave the action invariant. Specifically, we must have that the symmetry of the action over the constraint surface reads[11]

$$\begin{aligned} \delta_\epsilon q_m &= -(\bar{f}_{mn})^{-1}\frac{\partial \Omega^a}{\partial q_n}\epsilon_a \\ \delta_\epsilon \eta_a &= -\epsilon_a \end{aligned} \tag{13}$$

The ϵ_a form a set of infinitesimal parameters that characterize the transformations.

The point to be stressed here is that the symmetry transformations (13) may reflect either the gauge or reparametrization properties of an invariant theory, or they may as well be hidden symmetries of a noninvariant model. While the former is well described by Dirac's formalism using the first-class constraints as symmetry generators, the later has not an easy description in the usual canonical approach.

To give simple but nontrivial illustrations of how the sympletic formalism just described reveals a hidden symmetry, let us consider the examples of Floreanini-Jackiw chiral boson[27], and 2D Maxwell fields. The theory of self-dual bosonic fields proposed by Floreanini and Jackiw is described by the following action

$$S = \int d^2x(\dot{\phi}\phi' - \phi'\phi') \tag{14}$$

where dot and prime have their usual meaning as time and space derivatives. The equations of motion possess a left mover field solution, as it should, but in addition to that there is a space independent freedom in terms of an arbitrary function of time $h(t)$ as

$$\phi(x,t) = \phi_+(x+t) + h(t) \tag{15}$$

In terms of Dirac's formalism, this theory has only a single second-class constraint, being therefore unable to disclose the origin of the symmetry (15). The only hint for its presence comes from the fact that Dirac's constraint matrix does not have a unique inverse. On the other hand, the model's sympletic matrix,

$$f(x-y) = 2\partial_x \delta(x-y) \tag{16}$$

has as its only zero-mode, an arbitrary function of time which, according to (13), accounts for the symmetry (15) above. To fix this invariance one has to adjust the boundary conditions such that the sympletic matrix zero eigenvalue equation has no nontrivial solutions which, in this case can be obtained with chiral boundary conditions.

Another simple but nontrivial example is given by Maxwell electrodynamics in (1+1) space-time dimensions[28]. Following Dirac's approach one is bound to find two first-class constraints

$$\begin{aligned} \Omega_0 &= \pi_0 \\ \Omega_1 &= \partial_x E \end{aligned} \tag{17}$$

which are respectively the primary constraint and the secondary Gauss law constraint. Introducing the gauge fixing conditions

$$\begin{aligned} \chi_0 &= A_0 \\ \chi_1 &= \partial_x A \end{aligned} \tag{18}$$

corresponding to the radiation gauge, the constraints above will form a second-class set. If the boundary conditions of the problem are chosen such that the Laplacian operator (∂_x^2) has no nontrivial zero-eigenvalue solution, then the Dirac matrix of constraints will have a well defined and unique inverse. Certainly in such a gauge all

Dirac brackets vanish identically, indicating that no propagating modes survive after gauge fixing. This result could have been told in advance since the original phase-space having dimension four would result in a theory with zero degrees of freedom after four gauge conditions have been imposed. However, if the boundary conditions are less restrictive, there will be some (hidden) residual symmetry left over which, as we will see, is responsible for the theta-vacua structure displayed by the model. Let us consider the physical system above enclosed in a finite space-time box[29]:

$$S = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_{-\frac{R}{2}}^{\frac{R}{2}} dx \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) \tag{19}$$

The sympletic matrix is easily computed to be

$$f(x-y) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\partial_x \\ 0 & -\partial_x & 0 \end{pmatrix} \delta(x-y) \tag{20}$$

Here the sympletic variables are $\xi_k = \{A, E, \lambda\}$ with $A = A_1$, $A_0 = \dot{\lambda}$ and $E = F_{01}$. From (8) and (10) we get the zero-mode of (20) as

$$v(x) = \begin{pmatrix} \partial_x u \\ 0 \\ u \end{pmatrix} \tag{21}$$

with $u(x,t)$ being a totally arbitrary function of the space-time coordinates. This zero-mode displays the well known Abelian gauge symmetry of the model which reads

$$\begin{aligned} \delta A &= \partial_x u \\ \delta E &= 0 \\ \delta A_0 &= \partial_t u \end{aligned} \tag{22}$$

After imposing the radiation gauge, the sympletic matrix assumes the form

$$f_{rad}(x-y) = \begin{pmatrix} 0 & -1 & 0 & \partial_x \\ 1 & 0 & -\partial_x & 0 \\ 0 & -\partial_x & 0 & 0 \\ \partial_x & 0 & 0 & 0 \end{pmatrix} \delta(x-y) \tag{23}$$

This matrix displays the structure advanced in (8) with

$$\bar{f}(x-y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \delta(x-y) \tag{24}$$

and

$$\left(\frac{\partial \Omega}{\partial q}\right) = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} \delta(x-y) \tag{25}$$

and possess two zero-modes whose form is given by (10). For instance,

$$v^{(1)}(x) = \begin{pmatrix} \partial_x u \\ 0 \\ u \\ 0 \end{pmatrix} \tag{26}$$

with a similar one for the second zero-mode. As discussed above, the existence of a sympletic matrix zero-mode in this case will depend on the Laplacian equation $\partial_x^2 u = 0$ possessing a nontrivial solution. If the boundary conditions are chosen such that this is the case, then we have that the two Laplacian eigenvectors

$$\begin{aligned} u_1 &= c_1(t) \\ u_2 &= c_2(t)x \end{aligned} \tag{27}$$

do satisfy the following operator condition

$$D^{(k)} u_k = 0 \; ; \quad \text{(no sum)} \tag{28}$$

with

$$D^{(k)} = \{\partial_x, x\partial_x - 1\} \tag{29}$$

It is easy to check that these operators close an algebra under the commutator operation as

$$[D^{(k)}, D^{(m)}] = C_n^{km} D^{(n)} \tag{30}$$

where the only nonvanishing structure constants are $C_1^{12} = C_1^{21} = 1$. One can see that redefining these operators as

$$\begin{aligned} D^{(1)} \rightarrow \tilde{D}^{(1)} &= D^{(1)} \\ D^{(2)} \rightarrow \tilde{D}^{(2)} &= \frac{D^{(2)}}{D^{(1)}} \end{aligned} \tag{31}$$

the set $\tilde{D}^{(k)}$ will form a simple representation of the Heisenberg-Weyl algebra.

To see the physical consequences of this residual symmetry we follow Ref.[29] closely. Recall that the solution for the Laplacian equation reads

$$u(x,t) = a(t)x + b(t) \tag{32}$$

If one chooses $u(t, \frac{R}{2}) = u(t, -\frac{R}{2}) = 0$, then $u(x,t)$ vanishes identically and no symmetry is obtained. However, if one chooses instead $u(t, \frac{R}{2}) - u(t, -\frac{R}{2}) = 2\pi n$, then $u(t,x)$ is nontrivial and is given by

$$\begin{aligned} u(t,x) &= 2\pi \frac{x}{R} N(t) \\ N(t) &= (n_+ - n_-)\frac{t}{T} + \frac{1}{2}(n_+ + n_-) \end{aligned} \tag{33}$$

with $n_\pm$ being defined at the time boundary by

$$u_\pm(x) \equiv u(x, \pm\frac{T}{2}) = \frac{2\pi x}{R} n_\pm \tag{34}$$

In [29], the authors have called this solution at the time boundary as "classical vacua". The zero-mode of the sympletic matrix therefore describes the interpolation between two topologically different vacua, $u_+(x)$ and $u_-(x)$, with relative winding number $\nu = n_+ - n_-$. Observe that the action for the Maxwell field now reduces to that of a particle moving on a circle

$$S = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \frac{M}{2} \dot{N}^2 = \frac{1}{2} M \frac{(n_+ - n_-)^2}{T} \tag{35}$$

with the particle's mass parameter being proportional to the circle's inverse curvature $M = \frac{2\pi}{R}$. Here

$$\nu = n_+ - n_- = \frac{1}{2\pi} \int_{-\frac{R}{2}}^{\frac{R}{2}} dx \int_{-\frac{T}{2}}^{\frac{T}{2}} dt F_{01} \tag{36}$$

is the Pontryagin index and the zero-mode can safely be called as the instanton solution.

3 2D Induced Gravity

In 2D there is no Einstein equation since the Hilbert-Einstein action is a topological invariant measuring the genus of the manifold on which one integrates. Nevertheless, conformally invariant matter fields induce an action over the metric fields through the conformal anomaly. Let us now consider the action for the induced 2D gravity [13]

$$S = -k \int \sqrt{-g} R \Box^{-1} R d\tau d\sigma \tag{37}$$

with $k = c/96\pi$, and c being the central charge of the matter field. This effective action is nonlocal in the metric fields but the nonlocality can be removed introducing an auxiliary scalar field $\phi(\tau, \sigma)$[13, 30, 31]

$$S = -\frac{1}{2} \int \sqrt{-g} \left(g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi + \alpha R \phi \right) d\tau d\sigma \tag{38}$$

where α is a constant parameter. The coupling of the scalar curvature R with the auxiliary scalar field ϕ breaks the conformal invariance of the nonlocal model. To break the reparametrization invariance we must impose some gauge conditions. We choose to work in the light cone gauge characterized by $g_{\mu\nu} = \{g_{++} = 2h;\ g_{+-} = -1;\ g_{--} = 0\}$. With this choice the scalar curvature becomes

$$R = -2\partial_{x^-}^2 h \tag{39}$$

Using these results in (38) we obtain, after two integrations by parts

$$L = \int dx^- \left[\partial_{x^+} \phi \, \partial_{x^-} \phi + h(\partial_{x^-} \phi \, \partial_{x^-} \phi + \alpha \partial_{x^-}^2 \phi) \right] \tag{40}$$

As mentioned above, when described in instant-front variables[20], this model presents no constraints, while in light-cone variables only second-class constraints are present. This reflects the fact that all symmetries of the model have already been fixed. So, under Dirac's approach, this model should present no symmetries. To examine the existence of a (hidden) symmetry under the Faddeev-Jackiw point of view we have to construct the sympletic matrix for this model. To this end we first relabel the Lagrange multiplier field as $h \to \partial_{x^+} \lambda$, so that

$$L = \int dx^- \left[\partial_{x^+} \phi \, \partial_{x^-} \phi + \partial_{x^+} \lambda (\partial_{x^-} \phi \, \partial_{x^-} \phi + \alpha \partial_{x^-}^2 \phi) \right] \tag{41}$$

The sympletic matrix (in terms of ϕ and λ) reads

$$f(x,y) = \begin{pmatrix} -2\partial_{x^-} & \alpha\partial^2_{x^-} - 2\partial^2_{x^-}\phi - 2\partial_{x^-}\phi\partial_{x^-} \\ -\alpha\partial^2_{x^-} - 2\partial_{x^-}\phi\partial_{x^-} & 0 \end{pmatrix} \delta(x-y) \qquad (42)$$

One immediately notices the similar structure of this matrix with that in (8). Therefore, according to the discussion above, if there exist solutions to the matrix equation

$$\int dy^- f(x,y) v^{(a)}(y) = 0 \; ; \quad a = 1, \ldots, M \qquad (43)$$

then they must corresponds to the zero-modes of $f(x,y)$ that will reveal the symmetry remaining in the model. According to (10) the zero-modes must have the following structure

$$v^{(a)}(y) = \begin{pmatrix} \frac{1}{4}\int dz^- \epsilon(y^- - z^-) \left[\alpha\partial^2_{z^-} g_a(z) - 2\partial_{z^-}(g_a(z)\partial_{z^-}\phi)\right] \\ g_a(y) \end{pmatrix} \qquad (44)$$

which, after simple algebra, can be written in the following form

$$v^{(a)}(y) = \begin{pmatrix} \frac{1}{2}\alpha\partial_{y^-} g_a(y) - g_a(y)\partial_{y^-}\phi \\ g_a(y) \end{pmatrix} \qquad (45)$$

with $g_a(x)$ being a set of some yet undetermined functions. In components, the matrix equation (43) becomes the following pair of equations

$$\begin{aligned} 0 &= \int dy^- \partial_{x^-}\delta(x^- - y^-) \left(\alpha\partial_{y^-} g_a(y) - 2g_a(y)\partial_{y^-}\phi\right) \\ &\quad + \int dx^- \left(\alpha\partial^2_{x^-} - 2\partial^2_{x^-}\phi - 2\partial_{x^-}\phi\partial_{x^-}\right) \delta(x^- - y^-) g_a(y) \\ 0 &= \int dy^- \left(-2\partial_{x^-}\phi\partial_{x^-} + \alpha\partial^2_{y^-}\right) \delta(x^- - y^-) \left(\frac{\alpha}{2}\partial_{y^-} g_a(y) - g_a(y)\partial_{y^-}\phi\right) \end{aligned}$$

It is a simple algebra to check that the first equation is trivially satisfied, therefore imposing no restriction over the functional form of $g_a(x)$. The second equation, on the other hand, is only satisfied if

$$\alpha^2 \partial^3_{x^-} g_a(x) = 0 \; ; \alpha \neq 0 \qquad (46)$$

There are therefore three distinct null eigenvalues for the sympletic matrix (42) that reads

$$
\begin{aligned}
g_1(x) &= \epsilon^{(1)}(x^+) \\
g_2(x) &= \epsilon^{(2)}(x^+)x^- \\
g_3(x) &= \epsilon^{(3)}(x^+)(x^-)^2
\end{aligned} \tag{47}
$$

If one removes these three zero-modes (with properly chosen boundary conditions) then the sympletic matrix becomes nonsingular and the elements of the inverse give the Dirac brackets that quantize the theory. Observe that the three differential operators satisfying

$$
D^{(a)} g_a = 0 \ ; \ \text{(no sum)} \tag{48}
$$

are given by

$$
D^{(a)} = \{\partial_{x^-};\ x^-\partial_{x^-} - 1;\ (x^-)^2\partial_{x^-} - 2x^-\} \tag{49}
$$

that can be easily verified to be the generators of the SL(2,R) group algebra. Also, from (13) we obtain the symmetry transformation of the scalar field under this group of transformations as

$$
\begin{aligned}
\delta_\epsilon \phi &= -\left(\epsilon^{(1)}(x^+) + x^-\epsilon^{(2)}(x^+) + (x^-)^2\epsilon^{(3)}(x^+)\right)\partial_{x^-}\phi \\
&\quad + \frac{\alpha}{2}\left(\epsilon^{(2)}(x^+) + 2x^-\epsilon^{(3)}(x^+)\right)
\end{aligned} \tag{50}
$$

In conclusion, under the geometric approach of Faddeev-Jackiw, the SL(2,R) residual symmetry of Polyakov's model manifest itself in terms of the zero-modes of the sympletic matrix, just as any other explicit symmetry, such as gauge and reparametrization invariances. While the study of symmetries inside Dirac's formalism is based on the existence of first-class constraints, in the Faddeev-Jackiw formalism it relies on the orthogonality of the sympletic matrix zero-modes with the gradient of the (sympletic) potential. These (null) eigenvectors generate thus the isopotential lines of the theory, i.e. its symmetry lines, bypassing the first-class constraints for doing this job. This characterization of the Faddeev-Jackiw method in terms of null eigenvectors of the sympletic matrix, instead of constraints seems to be a definite advantage of this method over Dirac's formalism since it avoids the use of first-class constraints, which in cases as the one above are certainly not avaliable.

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Note Added: After the completion of our work we became aware of the Ref.[32, 33, 34] where these authors were able to recover Polyakov's 2D gravity using the coadjoint orbit method. In particular Alekseev and Shatashvili[32] have shown the existence of a SL(2,R) symmetry in the 2D gravity by relaxing the periodicity boundary condition. The main point of this method is to exploit the fact that a coadjoint orbit of a group admits a sympletic two-form that may be used to construct an action that enjoys at least as much symmetry as the group whose coadjoint orbit is under consideration. It should be noticed that the method of our paper, using the Faddeev-Jackiw formalism, follows the opposite pathway, in the sense that we use the zero-modes of a singular sympletic two-form to determine the symmetries enjoyed by the action that produced that particular singular sympletic form. It is also worth to point out that the coadjoint orbit method is a highy mathematical subject while our method is acessible to all.

References

[1] P.A.M.Dirac, Can.J.Math.2(1950)129, Lectures on Quantum Mechanics (Yeshiva University Press, New York, N.Y.) 1964.

[2] L.Faddeev, R.Jackiw, Phys. Rev. Lett. 60 (1988) 1692.

[3] C.Cronstrom and M.Noga, Nucl. Phys. B428 (1994) 449.

[4] O.V.Jacob, Phys.Rev. D50 (1994) 5289.

[5] J.W.Jun and C.Jue, Phys. Rev. D50 (1994) 2939.

[6] J.Barcelos-Neto and E.Cheb-Terrab, Z. Phys. C54 (1992) 133.

[7] D.S.Kulshreshtha and H.J.W.Muller-Kirsten, Phys. Rev. D43 (1991) 3376.

[8] M.M.Horta-Barreira and C.Wotzasek, Phys. Rev. D45 (1992) 1410.

[9] J.Barcelos-Neto and C.Wotzasek, Intl. J. Mod. Phys. A7 (1992) 4981.

[10] J.Barcelos-Neto and C.Wotzasek, Mod. Phys. Lett. A7 (1992) 1737.

[11] H.Montani and C.Wotzasek, Intl. J. Mod. Phys. A8 (1993) 3387.

[12] H.Montani, Intl. J. Mod. Phys. A8 (1993) 4319.

[13] A.M.Polyakov, Mod. Phys. Lett. A2 (1987) 893.

[14] V.G.Knizhnik, A.M.Polyakov, and A.B.Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.

[15] E.Abdalla, M.C.B.Abdalla, and K.D.Rothe, Non-Perturbative Methods in Two-Dimensional Quantum Field Theory (World Scientific, Singapore, 1991).

[16] E.Abdalla, M.C.B.Abdalla, J.Gamboa, and A.Zadra, Phys. Lett. B272 (1991) 222.

[17] E.S.Egorian and R.P.Manvelian, Mod. Phys. Lett. A5 (1990) 2371.

[18] J.A.Helayel-Neto, S.Mokhtari, and A.W.Smith, Phys. Lett. B236 (1990) 12.

[19] J.Barcelos-Neto, Phys. Rev. D49 (1994) 1012.

[20] P.A.M.Dirac, Rev. Mod. Phys. 21 (1949) 392.

[21] S.Ghosh and S.Mukherjee, Ann. Phys. (N.Y.) 231 (1994) 41.

[22] T.Regge and C.Teitelboin, Ann. Phys. (N.Y.) 88 (1974) 286.

[23] R.Benguria, P.Cordero, and C.Teitelboim, Nucl. Phys. B122 (1977) 61.

[24] M.Lledo and A.Restuccia, "On the canonical approach, the initial value problem and the Maximum Principle", USB/148/1990.

[25] J.Govaerts, Intl. J. Mod. Phys. A5 (1990) 3625.

[26] R.Jackiw, "Constrained Quantization Without Tears", CTP/2215, hep-th/9306075.

[27] R.Floreanini and R.Jackiw, Phys. Rev. Lett. 59 (1987) 1873.

[28] S.Coleman, Phys. Rev. D11 (1975) 2088.

[29] N.P.Ilieva and V.N.Pervushin, Sov. J. Part. Nucl. 22 (1991) 275.

[30] E.Abdalla, M.C.B.Abdalla, and A.Zadra, ICTP/89/56.

[31] K.Isler and J.M.Lina, Nucl. Phys. B358 (1991) 713.

[32] A.Alekseev and S.Shatashvili, Nucl. Phys. B323 (1989) 719.

[33] B.Rai and V.G.J.Rodgers, Nucl. Phys. B341 (1990) 119.

[34] G.W.Delius, P.van Nieuwenhuisen and V.G.J.Rodgers, Intl. J. Mod. Phys. A5 (1990) 3943.